# Notes on Mathematical Modelling

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## 0.1 Acknowledgements

These notes are based on the lectures of Carolin Kreisbeck (c.kreisbeck@uu.nl) during Fall of 2019 at Universiteit Utrecht. The lectures were intended to serve as a preparation for the reading of the texbook of the course [1].

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# Contents

# Chapter 1

# Basic tools for mathematical modelling

Teaching for this chapter started on Monday, 2019.11.11 (week 46a) and ended on Monday, 2019.11.18. This chapter corresponds to part of chapter 1 in [1].

In this introductory chapter we will introduce the mindset that we should have when trying to **translate a specific problem** from the natural sciences, the social sciences or technology into a **well-defined mathematical problem**[1].

TODO: some context and general pointers would probably look good here.

## 1.1 Case study: population dynamics

Suppose we want to model the change in population (i.e. number of individuals) in an environment over a period of time. First thing we need is to make some assumptions about what's really happening here. We might, for example, make the following assumptions[2]

1. growth rate independent of population size (unlimited growth possible, neglecting e.g. limited resources)
2. growth rate independent of time (neglecting time-dependence due to e.g. influence of enemies, economical or cultural changes)
3. population within closed systems (neglecting e.g. migration)
4. assuming an equal distribution of male and female, age distribution not considered
5. continuous model with non-integer solutions (idealization reasonable for very large populations, for small populations stochastic effects have to be taken into account)

[1] This is the definition of *mathematical modelling* given in [1, p. 1] with Kreisbeck's emphasis.

[2] Stolen from [2].

After this, we name the quantities that intervene in our problem. We will use $t$ for time, $x(t)$ for the number of individuals (population) at time $t$ and $\frac{dx}{dt}(t)$ or $x'(t)$ for the rate of change in population. To model the change we introduce the quantities

- $b(t, \Delta t)$ for the increase of population during the time interval $(t, \Delta t)$, and
- $d(t, \Delta t)$ for the decrease of population during the time interval $(t, \Delta t)$.

Therefore the population at time $t + \Delta t$ is given by

$$x(t + \Delta t) = x(t) + b(t, \Delta t) - d(t, \Delta t).$$

That $\Delta t$ desperately wants us to take the limit as $\Delta t \to 0$ and so we do

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{b(t, \Delta t)}{\Delta t} - \lim_{\Delta t \to 0} \frac{d(t, \Delta t)}{\Delta t}.$$

Note that here we're assuming the limit really does exist, which is quite a big assumption… Rename,

$$B(t) = \lim_{\Delta t \to 0} \frac{b(t, \Delta t)}{\Delta t} \text{ and } D(t) = \lim_{\Delta t \to 0} \frac{d(t, \Delta t)}{\Delta t}$$

and use the definition of derivative to get

$$\frac{dx}{dt}(t) = x'(t) = B(t) - D(t)$$

where $B(t)$ and $D(t)$ being the rates at which the population increasing, resp. decreases at time $t$. Recall that we assumed that the rates of change in the population were independent of time and population size. This is equivalent to saying that $B(t)$ and $D(t)$ are really constants which gives us the final model

$$x'(t) = \beta - \delta \implies x(t) = (\beta - \delta)t.$$

The previous is a not-particularly-interesting ODE with solution[3]

$$x(t) = (\beta - \delta)x + C.$$

This model has a lot of shortcomings, first of all, it does not account for the size of the population in the rates of change. But, one might argue that the more individuals there are in a population the greater the rates of change are. We can go back and restate assumption one as "population increase, resp. decrease in the time interval $(t, \Delta t)$ is directly proportional to the population at time $t$ and the time passed". This in turn gives us

$$b(t, \Delta t) = \beta x(t)\Delta t \text{ and } d(t, \Delta t) = \delta x(t)\Delta t.$$

Taking the limit as before leads us to the model

$$x'(t) = (\beta - \delta)x(t) = px(t).$$

[3]For help with solving differential equations see [3].

From now on, we shall let $p = \beta - \delta$ since we don't really need to distinguish between changes in the population because of births and deaths—we just care about the overall evolution of the population. This is another ODE, this time a bit more interesting, with solution

$$x(t) = Ce^{pt}.$$

Although a bit better, you can probably see that this model explodes as time passes since it does not include any provisions for when the population turns stupidly large. Anyhow, it is common enough that it deserves its own name:

the **exponential growth model**.

A small step in the right direction would be to account for a population limit in the system, i.e. number of individuals that flips the rate of growth. More precisely, let's change assumption one to "there is a number $x_M$ that is the maximum population in the system (sometimes called the *carrying capacity* and that the rate of change in population $p(x(t))$ is positive if $x(t) > x_M$ and negative if $x(t) > x_M$". The easiest way to model this is with a linear ansatz for $p(x)$, i.e. something of the form $p(x) = q(x_M - x)$ with a parameter[4] $q > 0$. Notice how

$$\begin{cases} p(x) > 0 & \text{if } x < x_M, \text{ and} \\ p(x) < 0 & \text{if } x > x_M \end{cases}.$$

Plugging this into our previous model to get

$$x'(t) = q(x_M - x(t))x(t) = qx_M x(t) - qx^2(t),$$

which is our final model for now and is called the **logistic growth model**

This ODE can be solved exactly and the solution is

$$x(t) = \frac{x_M x_0}{x_0 + (x_M - x_0)e^{-x_M q(t-t_0)}},$$

where $t_0$ is the initial time and $x_0 = x(t_0)$ is the initial population.

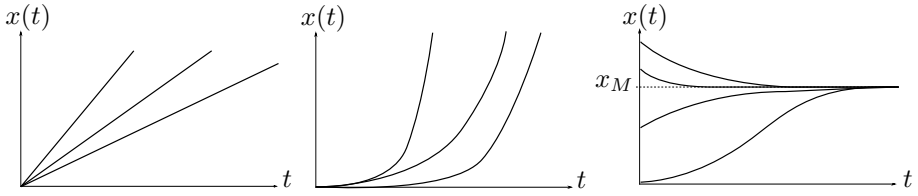


Figure 1.1: Solutions for the three iterations of the model, for different values for the parameters on each version.

We'll stop here for now, but keep in mind that we're missing the second half of the solution—we still need to apply this models to the real world. This

[4]The meaning of $q$ is a bit more complex to explain, but at heart it is just a proportionality constant.

means *fitting* those curves to the specific problem at hand, in this case, getting some data (at least two data points) to calculate the constants that are present in our solutions. Also, recall that we made a lot of assumptions, there are more population dynamics models that account for changes in the environment, migrations, etc.

## 1.2 Dimensional analysis and non-dimensionalisation

The previous models had two or three parameters each, but as we work our way to more complex examples the number of parameters will increase. Moving around all those constants is cumbersome and draws our attention away from really understanding the problem at hand. In addition, as we apply our models to specific problems we will need to take into account the units of the quantities we are dealing with.

### 1.2.1 Dimensional analysis

In addition to being a prerequisite to doing non-dimensionalisation, dimensional analysis provides a sanity check for us to *make sure we're not adding apples to oranges*. When coming up with a model, we generally need to specify the *physical dimension* of the quantities involved, but not necessarily want to specify the particular units that quantity is expressed in. For this we will denote by $[c]$ the physical dimension of a quantity $c$. For example if $t$ denotes time, when we write $[t]$ we mean the *dimension of time* or *some units of time* but do not specify which.

Revisiting our population model we can define the characteristic units

$$T := \text{ time }, \qquad \text{and} \qquad N := \#\text{ of individuals}$$

so that we get the following dimensions for the involved quantities

$$[t] = T,\ [x(t)] = N,\ [x'(t)] = \frac{N}{T}.$$

We can also do this for the parameters by solving for them in the equation for the model[5]

$$[x_M] = N,\ [t_0] = T,\ [x_0] = N,\ [q] = \frac{[x']}{[x_M - x][x]} = \frac{N/T}{N \cdot N} = \frac{1}{T}.$$

### 1.2.2 Non-dimensionalisation

Once we know the physical units of all involved quantities in our model we are ready to choose actual units for our model. For instance, for time we might choose years, days or hours but most of the time it is better to choose appropriate units for our problem. Non-dimensionalisation[6] is a recipe for choosing the most appropriate units, in the sense that we get the fewest parameters in our model. In essence, non-dimensionalisation is just a clever change of variables that leads

[5] Notice how $[x_M - x] = N$ and not something weird like $N - N = 0$, since when you subtract apples from apples you still get apples.

[6] Yes, this is an accepted spelling although not very common in the literature.

to a simplification of a model. In this section we describe a general procedure to find such changes of variables.

Let us begin with an example. From the dimensional analysis of our population dynamics examples we now that there are two physical dimensions and therefore we will choose two characteristic quantities $\bar{t}$ and $\bar{x}$. We use these to set up the following change of variables.

### 1.2.3 Non-dimensionalisation when there are several options. The projectile problem.

We now turn our attention to another problem—predicting the trajectory of a projectile. We take inspiration from Newton's works, specifically in Newton's second law

$$F = m \cdot a,$$

where $F$ is for force, $m$ for mass and $a$ is for acceleration, and in Newton's law of gravitation

$$F = -G\frac{Mm}{(x+R)^2},$$

where $G$ is the [empirical] gravitational constant, $M$ is the mas of the Earth, $m$ is the mass of the projectile, $x$ is the height of the projectile measured from the surface of the Earth and $R$ is the Earth's radius. Under the assumptions that there is no air resistance and that the Earth is a perfect sphere, where the mass is evenly distributed, Newton's law of gravitation is true. As is usual in physics parlance, we will introduce the quantity

$$g = \frac{GM}{R^2}$$

which is the force experienced by an object on the Earth's surface per unit mass, i.e. the gravitational acceleration at sea level.

This leads to the model

$$x''(t) = -G\frac{M}{(x+R)^2} = -\frac{gR^2}{(x+R^2)}, \qquad x(0) = 0, \qquad x'(0) = v_0,$$

where $v_0$ denotes the initial velocity.

We will now try to non-dimensionalise it. The involved quantities have dimensions $T$ for time and $L$ for length. The dimensions of the given data are

$$[v_0] = L/T, \qquad [R] = L, \qquad [g] = L/T^2.$$

The independent variable $t$ has dimension $[t] = T$ and the quantity we are after (height) has dimension $[x] = L$.

We introduce the change of variables

$$\tau = \frac{t}{\bar{t}} \text{ and } y = \frac{x}{\bar{x}},$$

where $\bar{t}$ and $\bar{x}$ are two characteristic quantities for time and length respectively and the new variables $\tau$ and $y$ are therefore dimensionless. The non-dimensionalised model is

$$y''(\tau) = -\frac{gR^2\bar{t}^2}{\bar{x}((\bar{x}/R)\cdot y(\tau)+R)^2}, \qquad y(0) = 0, \qquad y'(0) = \frac{\bar{t}}{\bar{x}}v_0,$$

or, equivalently,

$$y''(\tau) = -\frac{\bar{t}^2 g}{\bar{x}} \cdot \frac{1}{((\bar{x}/R) \cdot y(\tau) + 1)^2}, \qquad y(0) = 0, \qquad y'(0) = \frac{\bar{t}}{\bar{x}} v_0,$$

Now we want to choose $\bar{t}$ and $\bar{x}$ in such a way that as many of the appearing parameters as possible become equal to 1. Unfortunately, we have more parameters (3) than characteristic quantities (2), so we will need to make a compromise and only set two of the parameters equal to one (otherwise, the system of equations would have no solution). The three parameters are

$$p_1 = \frac{\bar{t}^2 g}{\bar{x}}, \qquad p_2 = \frac{\bar{x}}{R} \qquad \text{and} \qquad p_3 = \frac{\bar{t}}{\bar{x}} v_0.$$

If we set $p_1 = p_2 = 1$ we get $\bar{t} = \sqrt{R/g}$ and $\bar{x} = R$ and the model becomes

$$y''(\tau) = -\frac{1}{y(\tau) + 1)^2}, \qquad y(0) = 0, \qquad y'(0) = \sqrt{\frac{1}{gR}}.$$

If we set $p_1 = p_3 = 1$ we get $\bar{t} = v_0/g$ and $\bar{x} = v_0^2/g$ and the model becomes

$$y''(\tau) = -\frac{1}{(\frac{v_0^2}{gR} y(\tau) + 1)^2}, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Finally, if we set $p_2 = p_3 = 1$ we get $\bar{t} = R/v_0$ and $\bar{x} = R$ and the model becomes

$$y''(\tau) = -\frac{Rg}{v_0^2} \frac{1}{(y(\tau) + 1)^2}, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Although all the formulations are mathematically equivalent, we may prefer a formulation over another because of the ease of interpretation of the model, or for other reasons related to the motivation behind the model. In addition to these qualitative considerations, in the next section we will introduce a technique which will require models where the formulation has some specific characteristics. Therefore, the requirements of the technique will dictate the choice of non-dimensionalisation.

## 1.3 Asymptotic expansion method

### 1.3.1 Problems with small parameters

We will sometimes encounter models where the involvement of some quantity is very small but greatly complicates giving an answer to the question at hand. In those cases, we may be tempted to just approximate and disregard whatever part of the model has so little effect. Sometimes this situation will be inherent to the problem at hand but most of the time it will be due to our choice of quantities for the model, since in the former situation we normally disregard negligible effects when making the model assumptions.

Consider the task of finding the real solutions to the equation

$$x^5 + \varepsilon x - 1 = 0,$$

where $\varepsilon \ll 1$. There is no explicit formula for 5-th order polynomial equations but we would be able to solve it easily if we approximated $\varepsilon \approx 0$. For demonstration purposes, lets say that we want to find the real roots to

$$x^2 + 2\varepsilon x - 1 = 0.$$

We use a second order degree equation because we want to show that the method we are about to introduce gives a lot of precision with little work. Also, we introduce a factor of two in the equation because it makes the solution nicer when we use the quadratic formula.

Before moving on, let us introduce a couple of definitions.

**Definition 1.1** (Asymptotic sequence)**.** A sequence $(\phi_n(\varepsilon))_{n\in\mathbb{N}}$ is called an asymptotic sequence iff

$$\phi_{n+1}(\varepsilon) = o(\phi_n(\varepsilon)) \text{ as } \varepsilon \to 0.$$

An example of an asymptotic sequence is $\phi_n(\varepsilon) = \varepsilon^n$. This special case is called the asymptotic sequence in powers of $\varepsilon$.

**Definition 1.2** (Asymptotic expansion)**.** Let $(x_k)_{k\in\mathbb{N}}$ be any sequence and $(\varphi_n(\varepsilon))_{n\in\mathbb{N}}$ be an asymptotic sequence. The series

$$\sum_{k=0}^{N} \phi_k(\varepsilon) x_k$$

is called an asymptotic expansion of $x(\varepsilon)$ of the order $N \in \mathbb{N}\cup\{\infty\}$ with respect to the sequence $(\varphi_n(\varepsilon))_{n\in 355\mathbb{N}}$ if for $M = 0, \ldots, N$ we have

$$x(\varepsilon) - \sum_{k=0}^{M} \phi_k(\varepsilon) x_k = o(\phi_M(\varepsilon)) \text{ as } \varepsilon \to 0.$$

We will approximate the solution $x$ to the previous equation (1.3.1) with an asymptotic expansion of $x$ up to order 1 using powers of $\varepsilon$ as our asymptotic sequence. Formally, we write this as

$$x(\varepsilon) = x_0 + \varepsilon x_1 + O(\varepsilon^2).$$

Plugging this into (1.3.1) we get

$$x_0^2 + 2\varepsilon x_0 x_1 + \varepsilon^2 x_1^2 + 2\varepsilon(x_0 + \varepsilon x_1) - 1 = 0.$$

Since we said we only care about terms up to order 2 we can discard those containing $\varepsilon^2$ and write

$$x_0^2 + 2\varepsilon x_0 x_1 + 2\varepsilon x_0 + O(\varepsilon^2) - 1 = 0.$$

### 1.3.2 Error estimation

## 1.4 Exercises

**Exercise** (Recap, part a)**.** Solve the IVP

$$\begin{cases} y'(t) = \frac{2t}{y(t)-1}, \ t > 0 \\ y(0) = 2 \end{cases}$$

*Proof.* We ommit the $t$ in $y(t)$ and just write $y = y(t)$ for short. Via separation of variables we get

$$(y-1)y' = 2t.$$

Integrating on both sides with respect to $t$ we get,

$$\int (y-1)y' \ dt = \int 2t \ dt$$

and with the rule for substitution on integrals we rewrite it as

$$\int (y-1) \ dy = \int 2t \ dt.$$

Solving the integrals we arrive at the equality

$$\frac{y^2}{2} - y = t^2 + c$$

and solving for $y$ yields

$$y_{\pm} = 1 \pm \sqrt{1 - 2(t^2 + c)}.$$

Substituting the initial condition $y(0) = 2$ we get

$$1 \pm \sqrt{1 - 2c} = 2$$

which can only be satisfied with $y_+$ and picking $c = 0$. Therefore, the final solution to the IVP is

$$y(t) = 1 + \sqrt{1 - 2t^2}.$$

□

**Exercise** (Recap, part b)**.** Compute the solution to the second-order IVP

$$\begin{cases} y''(t) = y(t) + e^t, \ t > 0, \\ y(0) = 1, \ y'(0) = 1. \end{cases}$$

*Proof.*

1. Step 1: we find the general solution $y_h$ to the homogeneous ODE $y'' = y$. It follows from setting the characteristic polynomial $P(\lambda) = \lambda^2 - 1 = 0 \iff \lambda = \pm 1$. Therefore,

$$y_h(t) = c_1 e^t + c_2 e^{-t}.$$

2. Step 2: we find a particular solution $y_p(t)$ to the ODE $y'' = y + e^t$. One way to do it is with the method of indeterminate coefficients. We need a solution wich is independent from the homogeneous solution $y_h(t)$ so we make the ansatz
$$y_p(t) = ate^t.$$
Computing the first and second derivatives of $y_p(t)$ and substituting them into the original ODE (without boundary conditions) yields the equality
$$ae^t + ae^t + ate^t = ate^t + e^t \implies a = \frac{1}{2}.$$

3. Step 3: the solution to the IVP is given by
$$y(t) = y_h(t) + y_p(t) = c_1e^t + c_2e^{-t} + \frac{1}{2}e^t.$$
Pluging in the initial values for $y$ and $y'$ we arrive at the system of linear equations
$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 + \frac{1}{2} = 1 \end{cases} \iff \begin{cases} c_1 = \frac{3}{4} \\ c_2 = \frac{1}{4}. \end{cases}$$
Therefore, the solution to the IVP is given by the function
$$y(t) = \frac{3}{4}e^t + \frac{1}{4}e^{-t} + \frac{1}{2}te^t.$$

□

**Exercise** (1.4)**.** We consider the model of limited growth of populations
$$x'(t) = qx_Mx(t) - qx^2(t),\ x(0) = x_0.$$

1. Nondimensionalize the model using appropriate units for $t$ and $x$. Which possibilities exist?
2. What nondimensionalization is appropriate for $x_0 \ll x_M$ ($x_0$ "much smaller than" $x_M$) in the sense that omitting small terms leads to a reasonable model?

*Proof.* The two main involved quantities in this model are $t$ with dimension $T$ and $x(t)$ with dimension $N$. We make the following change of variables:
$$\tau = \frac{t}{\bar{t}}, \qquad y(\tau) = \frac{x(t) - x_0}{\overline{x}} = \frac{x(\tau\bar{t}) - x_0}{\overline{x}}.$$
We solve for $x(t)$ and $x'(t)$ to be able to plug $y(\tau)$ into the model:
$$x(t) = y(\tau)\overline{x}+x_0, x^2(t) = y^2(\tau)\overline{x}^2+x_0^2+2x_0\overline{x}y(\tau), x'(t) = \frac{d}{dt}\left(\overline{x}y(\tau) - x_0\right) = \overline{x}y'(\tau)\cdot\frac{1}{\bar{t}}.$$
Therefore,
$$\frac{\overline{x}}{\bar{t}}y'(\tau) = qx_M\left(\overline{x}y(\tau) + x_0\right) - q\left(\overline{x}^2y'(\tau) + x_0^2 + 2x_0\overline{x}y(\tau)\right),$$

or, equivalently,

$$y'(\tau) = qx_M\bar{t}y(\tau) + qx_M\frac{\bar{t}}{\overline{x}}x_0 - q\bar{t}\overline{x}y^t(\tau) - 2qx_0\bar{t}y(\tau).$$

As for the inital conditions we get

$$y(0) = \frac{x(0\cdot\bar{t}) - x_0}{\overline{x}} = 0$$

as expected, because of the way we chose the change $y(\tau)$.

There are many possibilities for the choice of $\bar{t}$ and $\overline{x}$. We will not discuss the now, but rather wait until part be. For an example of a detailed discussion without making extra assumptions see .

If we make the assumption that $x_0$ is very small, we may discard some terms in the above nondimensionalised model to get

$$\begin{cases} y'(\tau) = qx_M\bar{t}y(\tau) - q\bar{t}\overline{x}y^2(\tau), \\ y(0) = 0. \end{cases}$$

In this case, setting the cofficients for the function $y$ to equal 0 we get the system

$$\begin{cases} qx_M\bar{t} = 1, \\ x\bar{t}\overline{x} = 1 \end{cases} \iff \begin{cases} \bar{t} = \frac{1}{qx_M}, \\ \overline{x} = \frac{x_M}{\bar{t}}. \end{cases}$$

In this case there is only this posibility, since we have two restrictions on two characteristic quantities.

□

**Exercise** (1.5)**.** *(Nondimensionalization, scale analysis)* A body of mass $m$ is thrown upwards in a vertical direction from the Earth's surface with a velocity $v$. The air resistance is supposed to be taken into account by Stokes' law $F_R = -cv$ for the flow resistance in viscous fluids, which is reasonable for small velocities. Here $c$ is a coefficient depending on the shape and the size of the body. The motion is supposed to depend on the mass $m$, the velocity $v$, the gravitational acceleration $g$ and the friction coefficient $c$ with dimension $[c] = M/T$.

1. Determine the possible dimensionless parameters and reference values for height and time.

2. The initial value problem for the height is assumed to take the form

$$mx'' = cx' = -mg, \qquad x(0) = 0, \qquad x'(0) = v.$$

   Nondimensionalize the differential equation. Again different possibilities are available.

3. Discuss the different possibilities of a reduced model if $\beta := cv/(mg)$ is small.

*Proof.* TODO □

**Exercise** (1.6)**.** A model for the vertical throw on the Earth taking into account the air resistance is given by

$$mx''(t) = -mg - c\,|x'(t)|\,x'(t),\ x(t_0) = 0, x'(t_0) = v_0.$$

In this model the gravitational force is approximated by $F = -mg$, the air resistance for a given velocity $v$ is described by $-c\,|v|\,v$ with a proportionality constant $c$ depending on the shape and size of the body and the density of the air. This law is reasonable for high velocities.

1. Nondimensionalize the model. What possibilities exist?
2. Compute the maximal height of the throw for the data $m = 0.1\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, $v_0 = 10\,\mathrm{m/s}, c = 0.01\,\mathrm{kg/m}$ and compare the result with the corresponding result for the model without air resistance.

*Proof.* We make the change of variables

$$\tau = \frac{t - t_0}{\bar{t}}, \qquad y = \frac{x}{\bar{x}}$$

and plug it into the IVP to get

$$y''(\tau) = -\frac{\bar{t}^2 g}{\bar{x}} - \frac{c\bar{x}}{m}\,|y'(\tau)|\,y'(\tau), \qquad y(0) = 0, \qquad y'(0) = \frac{\bar{t}}{\bar{x}}v_0.$$

We would like as many coefficients as possible to equal 1 but we have 3 coefficients and 2 characteristic quantities so we will have to make a compromise. There are three possibilities for the compromise:

1. Set $\frac{\bar{t}^2 g}{\bar{x}} = \frac{c\bar{x}}{m} = 1$ and leave $\frac{\bar{t}}{\bar{x}}v_0$ as is, which would yield $\bar{t} = \sqrt{\frac{m}{gc}}$, $\bar{x} = \frac{m}{c}$, and
$$y'' = -1 - |y'|\,y', \qquad y(0) = 0, \qquad y'(0) = v_0\sqrt{\frac{c}{gm}}.$$
2. Set $\frac{\bar{t}^2 g}{\bar{x}} = \bar{t}/\bar{x}v_0 = 1$ and leave $\frac{c\bar{x}}{m}$ as is, which would yield $\bar{t} = \frac{v_0}{g}$, $\bar{x} = \frac{v_0^2}{g}$, and
$$y'' = -1 - \frac{cv_0^2}{mg} - |y'|\,y', \qquad y(0) = 0, \qquad y'(0) = 1.$$
3. Set $\frac{c\bar{x}}{m} = \bar{t}/\bar{x}v_0 = 1$ and leave $\frac{\bar{t}^2 g}{\bar{x}}$ as is, which would yield $\bar{t} = \frac{m}{cv_0}, \bar{x} = \frac{m}{c}$, and
$$y'' = -\frac{mg}{cv_0^2} - |y'|\,y, \qquad y(0) = 0, \qquad y'(0) = 1.$$

The second part is left to the reader. □

**Exercise** (1.8)**.** *(Formal aymptotic expansion)*

1. For the initial value problem
$$x''(t) + \varepsilon x'(t) = -1, \qquad x(0) = 0, \qquad x'(0) = 1$$
compute the formal asymptotic expansion of the solution $x(t)$ up to the second order in $\varepsilon$.

2. Compute the formal asymptotic expansion for the instance of time $t^* > 0$, for which $x(t^*) = 0$ holds true, up to first order in $\varepsilon$, by substituting the series expansion $t^* \sim t_0 + \varepsilon t_1 + O(\varepsilon^2)$ into the approximation obtained for $x$ leading to a determination of $t_0$ and $t_1$.

*Proof.* We make the ansatz

$$x_\varepsilon(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)$$

and plug it (we ommit the dependence on $t$ for brievity) into the IVP to get

$$x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \varepsilon \left(x_0' + \varepsilon x_1' + \varepsilon^2 x_2'\right) + O(\varepsilon^3) = -1$$

Collecting terms in $\varepsilon$ and discarding higher order terms yields

$$\begin{cases} O(\varepsilon^0): & x_0'' + 1 = 0 \\ O(\varepsilon^1): & x_1'' + x_0 = 0\,. \\ O(\varepsilon^2): & x_2'' + x_1' = 0 \end{cases}$$

For the first equation we integrate twice to get

$$x_0(t) = -\iint 1\ dt = -\int t + c_1\ dt = -\frac{1}{2}t^2 + c_1 t + c_0.$$

Since we know $x(0) = 0$ and $x'(0) = 1$ we deduce the values of the constants to get

$$x_0(t) = -\frac{1}{2}t^2 + t.$$

Plugging this into the second equation ($\varepsilon$ order 1) yields

$$x_1''(t) = \frac{1}{2}t^2 - t.$$

Integrating twice yields

$$x_1(t) = \frac{1}{24}t^4 - \frac{1}{6}t^3 + c_1 t + c_0.$$

Again, using the boundary conditions we find $c_1 = 9/8$ and $c_0 = 0$ and conclude

$$x_1(t) = \frac{1}{24}t^4 - \frac{1}{6}t^3 + \frac{9}{8}t.$$

Plugging this into the third equation we get

$$x_2''(t) = -\frac{1}{6}t^3 + \frac{1}{2}t^2 - \frac{9}{8}.$$

Integrating twice yields

$$x_2(t) = -\frac{1}{120}t^5 + \frac{1}{24}t^4 - \frac{9}{16}t^2 + c_1 t + c_0.$$

Again, using the boundary conditions we find $c_1 = 127/240$ and $c_0 = 0$.

TODO something must be wrong here □

**Exercise** (1.9)**.** A model already nondimensionalised for the vertical throw with *small* air resistance is given by

$$x''(t) = -1 - \varepsilon(x'(t))^2, \qquad x(0) = 0, \qquad x'(0) = 1.$$

The model describes the throw up to the maximal height.

1. Compute the first two coefficientes $x_0(t)$ and $x_1(t)$ in the asymptotic expansion
$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$$
for small $\varepsilon$.

2. Compute the maximal height of the throw up to terms of order $\varepsilon$ using asymptotic expansion.

3. Compare the results from 2. for the data of  with the exact result and the result neglecting the air resistance.

*Proof.* Informally (i.e. without carrying the $O(\varepsilon^2)$ term around), we get

$$x_0'' + \varepsilon x_1'' = -1 - \varepsilon(x_0' + \varepsilon x_1')^2.$$

Collecting terms in $\varepsilon$ we get

$$\begin{cases} x_0'' + 1 = 0 \\ x_1'' - x_0'^2 = 0 \end{cases}.$$

Solving the first equation together with the boundary conditions yields

$$x_0(t) = -\frac{1}{2}t^2 + t.$$

Plug $x_0$ into the second equation and use the boundary conditions to get

$$x_1(t) = \frac{1}{12}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + t.$$

Therefore the solution is

$$x_\varepsilon(t) = -\frac{1}{2}t^2 + t + \varepsilon\left(\frac{1}{12}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + t\right).$$

□

**Exercise** (1.10)**.** *(Multiscale approach)* The function $y(t)$ is supposed to solve the initial value problem

$$y''(t) + 2\varepsilon y'(t) + (1 + \varepsilon^2)y(t) = 0, \qquad y(0) = 0, \qquad y'(0) = 1,$$

for $t > 0$ and a small parameter $\varepsilon > 0$.

1. Compute the approximation of the solution by means of formal asymptotic expansion up to first order in $\varepsilon$.

2. Compare the function obtained in 1. with the exact solution

$$y(t) = e^{-\varepsilon t} \sin t.$$

For which times $t$ the approximation from 1. is good?

3. To get a better approximation one can try the approach

$$y \sim y_0(t, \tau) + \varepsilon y_1(t, \tau) + \varepsilon^2 y_2(t, \tau) + \dots,$$

here $\tau = \varepsilon t$ is a slow time scale. Substitute this ansatz in the differential equation and compute $y_0$ such that the approximation becomes better. *Hint:* The equation of lowest order does not determine $y_0$ uniquely and coefficient functions in $\tau$ apper. Choose them in a clever way such that $y_1$ is easily computable.

*Proof.* TODO □

# Chapter 2

# Linear systems of equations

## 2.1 Modelling electrical netwroks

# Chapter 3

# Ordinary differential equations

Teaching started on Monday 2019.11.25 (week 48a).

This chapter corresponds to part of chapter 4 in [1].

## 3.1 Quantitative analysis of models in population dynamics

Recall from week 46 (chapter 1) that we had two models for population dynamics.

- The first one, the exponential model was described by

$$x'(t) = px(t), \quad p \in \mathbb{R},$$

  where $p$ was the growth rate.

- The second, the constrained model was described by

$$x'(t) = qx_M x(t) - qx^2(t), \quad q, x_M \in \mathbb{R}$$

  where $q > 0$ was the growth rate and $x_M$ was the maximum carrying capacity of the environment in number of individuals.

Both of these models share the common mathematical structure of an autonomous equation, i.e. an equation of the form

$$x'(t) = f(x(t)), \tag{3.1}$$

where $f$ (read $x'$) does not depend explicitly[1] on $t$.

In this section we will focus on the qualitative aspects of the model, i.e. what information can we get from it without explicitly solving the equations (which in this case we can, but in the next examples we won't).

[1] That is, $f$ cannot *unwrap* $t$ out of $x(t)$ and do anything with it alone, it has to work on $x(t)$ as its variable.

Recall that a **stationary solution** of an ODE is one that stays constant in time, i.e. of the form $x(t) = c$. How can we find them? Easy, if $x$ is constant then we must have $x'(t) = f(x(t)) = 0$. For our previous models this means

- $x(t) = 0$ for the exponential model, and
- $x(t) = 0$ or $x(t) = x_M$ for the second model. We get these two solutions from solving
$$x'(t) = qx_M x(t) - qx^(t) = 0$$
for $x(t)$ using the well known quadratic formula.

We are interested in these solutions because they are predictable and *don't blow up* as time passes. Later in this chapter we will formally define the concept of stability and quantify how stable solutions are based on how close they are to the stationary solutions.

### 3.1.1 Introduction to linear stability analysis

For now we will settle with something called **linear stability analysis**. The main idea is to linearise the solution (i.e. Taylor expand up to degree 1) a stationary solution. Let $x^*$ be a stationary solution to an autonomous problem of the form 3.1. The linear expansion we are talking about is

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + O(|x - x^*|).$$

To make things easier, let us take $y(t) = x(t) - x^*(t)$. We shall ignore the error term $O(|x - x^*|) = O(y(t))$ and thus we get

$$y'(t) = f'(x^*)y(t). \tag{3.2}$$

Now, 3.2 is trivial to solve explicitly—it is a linear homogeneous equation with constant coefficients

$$y(t) = ce^{f'(x^*)t}.$$

Intuitively, as $t \to \infty$ we have

$$|y(t)| \to 0 \implies |x(t) - x^*(t)| \to 0 \iff x(t) \to x^*(t),$$

i.e. the linearised solution $y(t)$ converges to the stationary solution $x^*(t)$.

More on this later.

## 3.2 Predator–prey models

### 3.2.1 Derivation of the Lotka–Volterra equations

Now we turn our attention to environments where there are two species and one eats/hunts/harvests the other. Let us model this from scratch to get yet another example of how things work in Mathematical modelling. For this derivation we shall use the following assumptions.

1. The prey population has unlimited resources available for its growth all the time.
2. The predator population feeds exclusively on the prey population.
3. TODO: Something i cant remember.
4. The rate of growth of the populations is proportional to their size.
5. The environment is stable over time.

We now proceed with the standard recipe for deriving models.

1. **Name the quantities involved in the problem.** We are trying to model how two populations change over time so we need

$$\begin{aligned} t &:= \text{time} \\ x_1(t) &:= \text{size of prey population at time } t \\ x_2(t) &:= \text{size of predator population at time } t \end{aligned}$$

2. **Find relations between the quantities.** From assumption X we now that the growth of both species is directly proportional to the size of the populations, in other words

$$x_1' = p_1 x_1 \text{ and } x_2' = p_2 x_2.$$

A priori, we don't know if $p_1$ depends only on $t$ or on $x_2(t)$ or on both. The possibility that $p_1$ depends on $x_1(t)$ is ruled out by the assumption that growth is proportional to size. Looking at the assumptions once more we find that $p_1$ cannot depend on $t$ since "the environment is stable over time". There fore it must be that $p_1$ is a function only of $x_2(t)$, which really makes sense, since the size of the prey species depends on how many individuals are being eaten by the predator species. A similar argument for $p_2$ yields

$$p_1(x_2(t)) \text{ and } p_2(x_1(t)).$$

But what do these functions $p_1$ and $p_2$ look like? Well, the prey population naturally grows since we assumed unlimited resources but at the same time it is being eaten at some rate by the predator population. Similarly, the predator population naturally dies unless they can feed on the prey population. We introduce the parameters $\alpha, \beta, \gamma, \delta > 0$ and formalise these relations with

$$p_1(x_2(t)) = -\beta x_2(t) + \alpha \qquad \text{and} \qquad p_2(x_1(t)) = \delta x_1(t) - \gamma.$$

Finally we get our model, commonly referred to as the Lotka–Volterra equations, derived independently by both authors from around 1920 to around 1925 [4].

$$\begin{cases} x_1' &= (\alpha - \beta x_2) x_1 \\ x_2' &= (\delta x_1 - \gamma) x_2 \end{cases}. \tag{3.3}$$

The mathematical structure of this problem is that of an planar system of autonomous ODEs. We may rewrite it as

$$\begin{cases} \mathbf{x}' & = f(\mathbf{x}) \\ \mathbf{x}(t_0) & = \mathbf{x_0} \end{cases} \tag{3.4}$$

with $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. If $f$ is sufficiently nice (i.e. locally Lipschitz) then an initial value problem of the form in 3.4 has locally unique solutions. However, it is not the explicit solutions that interest us right now, but rather the qualitative aspects of their behaviour.

### 3.2.2 Qualitative analysis of the Lotka–Volterra equations

We look into the stationary solutions to later look at stability. Once more, setting $x_1, x_2 = 0$ in 3.3 gives us the stationary solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha/\beta \\ \gamma/\delta \end{pmatrix}.$$

There are too many parameters to work comfortably with this solutions. Let us non-dimensionalise before moving on to get rid of as many parameters as we can. Very quickly, we choose the fundamental dimensions $T$ for time and $N$ for number of individuals and carry out a dimensional analysis to get

$$\begin{array}{rlrl} [t] & = T & [x_1] = [x_2] & = N \\ [\alpha] & = [\gamma] = \frac{1}{T} & [\beta] = [\delta] & = \frac{1}{NT} \end{array}.$$

We choose the characteristic quantities $\overline{x_1}, \overline{x_2}$ and $\bar{t}$ and set up the change of variables

$$z_1 = \frac{x_1}{\overline{x_1}}, \qquad z_2 = \frac{x_2}{\overline{x_2}} \qquad \text{and} \qquad \tau = \frac{t}{\bar{t}}.$$

Substitute with care in 3.3 (careful with the derivatives) to get

$$\begin{cases} z_1' & = \bar{t}\alpha z_1 - \beta\overline{x_2}\bar{t} z_1 z_2 \\ z_2' & = \delta\overline{x_1}\bar{t} z_1 z_2 - \gamma\bar{t} z_2 \end{cases}.$$

Notice how we have four different coefficients for $z_1$ and $z_2$ but only have three characteristic quantities. This means we'll need to make a compromise. Which one to make is dictated by our taste and the mathematical or biological interpretation of the parameters we choose. We will not do all four options here but the Lotka–Volterra equations often come with

$$\bar{t}\alpha = 1, \qquad \beta\overline{x_2}\bar{t} = 1 \qquad \text{and} \qquad \delta\overline{x_1}\bar{t} = \gamma\bar{t},$$

which in turn give us the **non-dimensionalised version of the Lotka–Volterra equations**

$$\begin{cases} z_1' & = (1 - z_2) z_1 \\ z_2' & = a(z_1 - 1) z_2 \end{cases}, \tag{3.5}$$

where there is only one parameter $a = \gamma/\alpha$.

In this form, the stationary solutions are given by

$$\mathbf{z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

**Directional field of the Lotka-Volterra equations**

Plotting the directional field is a tool that proves useful when we want to get an overall idea of how the system behaves. Recall that given an autonomous ODE of the form $\mathbf{x}' = f(\mathbf{x})$ the directional field is a vector field of the form $x \mapsto f(x)$. In two dimensions this corresponds to a graph where the axes represent the values that the functions $x_1$ and $x_2$ can take and the arrows point in the direction $(f_1(x), f_2(x))$.

It is hard to draw these diagrams by hand but there are a number of steps we can take to get an idea of what they look like. Let's go back to the Lotka-Volterra model in the non-dimensionalised form. We plot the directional field by following these steps

1. **Find the stationery solutions.** These become points in the vector field since $f(x) = 0$ by definition of stationary solution (and therefore there is no arrow to draw).

2. **Find the isoclines**, i.e. the curves[2]

$$N_1 = \{f_1 = 0\} = \{\mathbf{z} \mid f_1(\mathbf{z}) = 0\} = \{(z_1, z_2) \mid f_1(z_1, z_2) = 0\}$$
$$N_2 = \{f_2 = 0\} = \{\mathbf{z} \mid f_2(\mathbf{z}) = 0\} = \{(z_1, z_2) \mid f_2(z_1, z_2) = 0\}$$

3. **Find the areas of monotonicity**, i.e. the regions of the plane given by[3]

$$D_{++} = \{f_1 > 0\} \cap \{f_2 > 0\}$$
$$D_{+-} = \{f_1 > 0\} \cap \{f_2 < 0\}$$
$$D_{-+} = \{f_1 < 0\} \cap \{f_2 > 0\}$$
$$D_{--} = \{f_1 < 0\} \cap \{f_2 < 0\}$$

   Keep in mind that the isoclines tell us where there is a change from one area of monotonicity to another[4].

4. **Draw the vector field**, i.e. some arrows.

[2]Here we mean *curve* in the most general sense—isoclines can be straight lines or curves and they can be disjoint as is the case in this example.

[3]It is useful to write them as an intersection since therefore we can reuse the sets $\{f_n < 0\}$ and $\{f_n > 0\}$ to calculate all the regions depending on how they overlap.

[4]The reason for this is that we normally ask that the functions involved in the ODE are *nice enough*, i.e. at least $C^2$ which means that the derivatives are continuous.

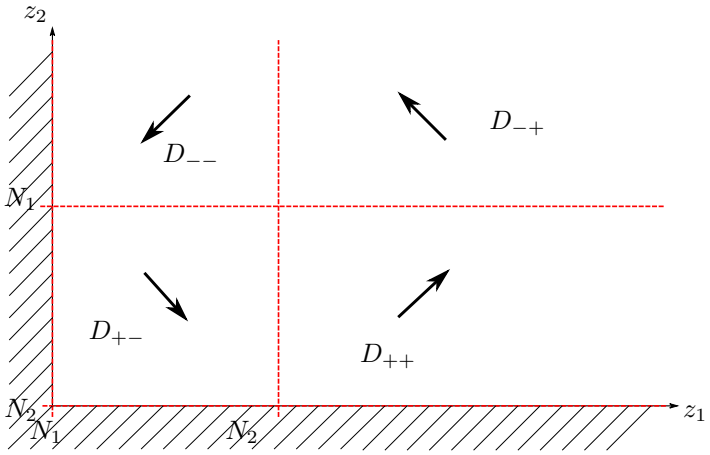


Figure 3.1: Sketch of the direction field for the Lotka-Volterra equations.

In the computer generated picture it is probably easier to tell what is happening with the solutions. If one were to choose a point in the plane, a solution going through that point would move in the direction of the vector in that point. Repeating this process can give us a rough idea of what the solutions look like: orbits around the stationary solution $(1,\ 1)$.

### 3.2.3 Phase diagram of the Lotka-Volterra equations

In our qualitative analysis of the Lotka-Volterra model we have found the stationary solutions and more or less characterised the rest of the solutions using the direction field. We would, however, like to have a more precise idea of what the solutions look like. We will devote the rest of this section to that.

**Definition 3.1** (First integral)**.** TODO. we really need a definition here :(

## 3.3 Stability theory

Consider the general autonomous system

$$\mathbf{x}' = f(\mathbf{x}) \tag{3.6}$$

where $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n,\ \Omega$ is open and $f \in C^2(\Omega)$.

**Definition 3.2** (Lyapunov stability)**.** Let $\mathbf{x}^*$ be a stationary solution to 3.6. We say $\mathbf{x}^*$ is Lyapunov stable (or just stable for short) if for every open neighbourhood $U$ of $\mathbf{x}^*$ there exists an open neighbourhood $V$ of $\mathbf{x}^*$ such that any solution $\mathbf{x}$ of 3.6 with $\mathbf{x}(0) \in B$ satisfies $\mathbf{x}(t) \in U$ for all $t > 0$.

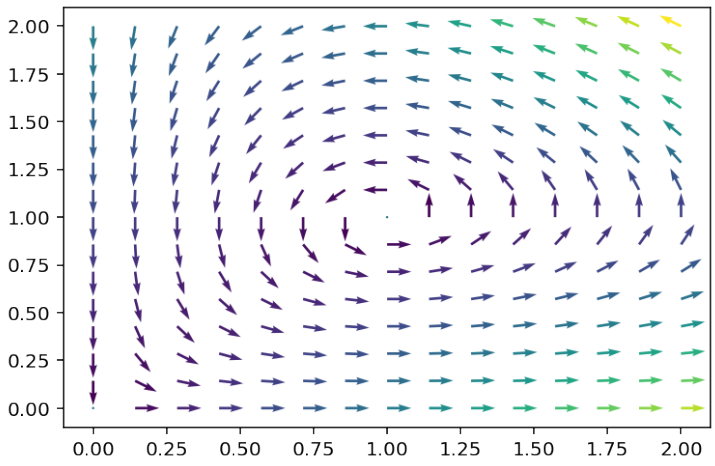


Figure 3.2: Direction field for the Lotka-Volterra model with $a = 0.5$ generated by a computer.

**Definition 3.3** (Asymptotic stability)**.** Let $\mathbf{x}^*$ be a stationary solution to 3.6. We say $\mathbf{x}^*$ is asymptotically stable if there exists an open neighbourhood $W$ of $\mathbf{x}^*$ such that for any solution $\mathbf{x}$ of 3.6 with $\mathbf{x}(0) \in W$ it holds that

$$\|\mathbf{x}(t) - \mathbf{x}^*\| \to 0 \text{ as } t \to \infty.$$

**Remark 3.4.** If $\mathbf{x}^*$ is asymptotically stable then it is Lyapunov stable.

### 3.3.1 Linear stability analysis

**Definition 3.5.** Let $\mathbf{x}^*$ be a stationary solution to 3.6. The linear system

$$\mathbf{y}' = Df(\mathbf{x}^*)\mathbf{y} \tag{3.7}$$

is called a linearisation of 3.6 in $\mathbf{x}^*$.

Moreover, $\mathbf{x}^*$ is called linearly unstable, resp. stable / asymptotically stable if $\mathbf{0}$ is unstable, resp. stable / asymptotically stable for .

**Theorem 3.6** (Principle of linearised stability)**.** If $\mathbf{x}^*$ is linearly asymptotically stable, resp. unstable, then $\mathbf{x}^*$ is asymptotically stable, resp. unstable.

**Remark 3.7.** The previous theorem does not work for Lyapunov stability, i.e.

$$\mathbf{x}^* \text{ linearly stable } \not\Rightarrow \mathbf{x}* \text{ stable.}$$

# Chapter 4

# Calculus of variations

**Definition 4.1** (Variational problem)**.** Let $X$ be a real vector space (of functions, possibly infinite dimensional), $\mathcal{A} \subset X$ a set of admissible functions and $\mathcal{I} : X \to \mathbb{R}$ a functional that assigns a real number for each $u \in X$.

A variational problem is the task

$$\text{minimise } \mathcal{I}(u), \quad u \in \mathcal{A}.$$

In particular we are concerned with variational problems of integral form, i.e. those where

$$\mathcal{I}(u) = \int_a^b f(x, u(x), u'(x))dx, \tag{4.1}$$

where $u : [a, b] \to \mathbb{R}^m$ and $f : [a, b] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$.

As mathematicians, we are immediately concerned about the existence and uniqueness of solutions. Until 1850, it was thought that minimisers for integral problems always existed but Weierstrass gave a counter example. From that moment on, a new theory for solving variational problems, now known as the **direct method** was developed. In this section we will mostly concentrate on the classical or **indirect method** for finding minimisers which basically replicates the process of finding minima for functions in vector calculus. The reason for this is that a formal treatment of the direct method requires advanced mathematical tools from functional analysis, which is not a prerequisite for this course.

**A word about notation.** As usual during this course, notation is a mess. Now we are using the notation $u'(x)$ to represent the derivative of a vector-valued one-variable function $u : \mathbb{R} \to \mathbb{R}^m$. In formal analysis one would write this as

$$Du(x) = \begin{pmatrix} \partial_x u_1(x) \\ \partial_x u_2(x) \\ \vdots \\ \partial_x u_m(x) \end{pmatrix} = \begin{pmatrix} \frac{d}{dx} u_1(x) \\ \frac{d}{dx} u_2(x) \\ \vdots \\ u_m(x) \end{pmatrix}.$$

I think it may be the heavy influence of geometry in this field that leads us to use the *prime* notation as is normally in analytic geometry.

Also, as usual, the notion of row and column vectors is a bit blurry and we will just assume that they have the right shape as needed. Clarification will be provided when the notation is unclear.

## 4.1 The indirect method

This method was developed by Euler and Lagrange during the 18th century. The strategy for dealing with variational problems of the form (4.1) is to just go ahead and find the minima of the function $\mathcal{I}(u)$. For this, let us recall the approach taken in vector calculus to minimise a function $f : \mathbb{R}^m \to \mathbb{R}^n$.

1. Find $\overline{x} \in \mathbb{R}^m$ such that $Df(\overline{x}) = \mathbf{0}$,
2. Find $D^2 f(\overline{x})$, and
3. Check that $D^2 f(\overline{x})$ is positive definite.

The problem with this strategy is that we don't know if $Df$ or $D^2 f$ will exist for our functional $f = \mathcal{I}$. Therefore, we will introduce a weaker notion of derivative, the variation, which is easier to work with in the context of these problems.

**Warning.** Scrutinise the results given in this chapter regarding the necessary and sufficient conditions for the existence of local minimisers. In particular, a positive second derivative on a critical point does not imply that critical point is a local minimiser. See example TODO and theorem TODO.

**Definition 4.2** (Variation, general definition)**.** Let $X$ be a function space and $J : X \to \mathbb{R}$ be a functional over $X$. We define the $k$-th variation of the functional $J$ at $u \in X$ in the direction of $\varphi \in X$ as the limit

$$\delta^k J(u)(\varphi) = \frac{d^k}{d\varepsilon^k} J(y + \varepsilon\varphi)\bigg|_{\varepsilon=0}.$$

In our setting, where we only want to consider admissible functions, we use a version of the definition of variation which incorporates this restriction.

**Definition 4.3** (Admissible perturbation)**.** Let $X$ and $\mathcal{A}$ be a function space and a set of admissible functions, resp. Let $u \in \mathcal{A}$. For any $\varphi \in X$ we say $\varphi$ is an admissible perturvation of $u$ iff there exists a $\varepsilon_0 > 0$ such that

$$u + \varepsilon\varphi \in A, \quad \text{for every } \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

**Definition 4.4** (Variation)**.** Let $X$ be a function space and let $\mathcal{A} \subset X$ be the set of admissible functions. We define the $k$-th variation of a functional $\mathcal{I}$ over $X$ at $u \in \mathcal{A}$ in the direction of an admissible perturbation $\varphi$ as the limit

$$\delta\mathcal{I}(u)(\varphi) := \frac{d^k}{d\varepsilon^k} \mathcal{I}(u + \varepsilon\varphi)\bigg|_{\varepsilon=0}. \tag{4.2}$$

Finally, we recall the definition of a local minimiser.

**Definition 4.5** (Local minimiser)**.** Let $X$ be a function space and $\mathcal{I} : X \to \mathbb{R}$ be a functional over $X$. We say $\overline{u} \in X$ is a local minimiser of $\mathcal{I}$ iff there exists some $\delta > 0$ such that

$$\forall u \in X,\ \|u - \overline{u}\|_X < \delta \implies \mathcal{I}(u) \geq \mathcal{I}(\overline{u}).$$

In the previous definition one may use any norm defined over $X$ to compute the distance between $u$ and $\overline{u}$. For function spaces like $X = \{f : A \to B\}$ it is common to choose the supremum norm defined by

$$\|f\|_X = \sup_{x \in A} f(x).$$

Now we use these definitions to derive necessary, *and only necessary*, conditions for the existence of local minimisers.

**Theorem 4.6** (Necessary conditions for local minimisers)**.** Let $X$ be a function space and $\mathcal{A} \subset X$ be a set of admissible functions. Let $\overline{u} \in \mathcal{A}$ be a local minimiser of the functional $\mathcal{I}$. Then,

1. $\delta\mathcal{I}(\overline{u})(\varphi) = 0$ for every admissible perturbation $\varphi$, and
2. $\delta^2\mathcal{I}(\overline{u})(\varphi) > 0$ for every admissible perturbation $\varphi$.

*Proof.* Define $g : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ be defined as $g(\varepsilon) = \mathcal{I}(\overline{u} + \varepsilon\varphi)$. Then 0 is a local minimiser for $g$. Hence $g'(0) = 0$ and $g''(0) = 0$. □

If you are curious, the reason for theorem 4.6 not providing necessary conditions is that the dimension of the functions space $X$ may be infinite and then things fail. Keep in mind that this theorem can only be used for two things

- Proving that a given $\overline{u}$ is not a local minimiser if it does not fulfill conditions 1 and 2 in theorem 4.6 (using the negated reciprocal), and
- Finding candidates for functions $u \in \mathcal{A}$ to be local minimisers. This is not very useful.

Also keep in mind that if one wants to prove that a function which satisfies the hypotheses in theorem 4.6 is not a local minimiser one cannot use the theorem but rather has to come up with something else.

### 4.1.1 The Euler-Lagrange theorem

Now we go back to the kinds of problems we stated at the beginning of the chapter: variational problems of integral form. Let us revisit the setting.

Consider a variational problem of the form

$$\text{minimise } \mathcal{I}(u) = \int_a^b f(x, u(x), u'(x))\ dx, \tag{4.3}$$

where $u \in \mathcal{A} = \{u \in X \mid u(a) = u_a,\ u(b) = u_b\}$ and $X = C^1([a, b], \mathbb{R}^m)$ and thus $f : [a, b] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. Additionally, for notation purposes, let us assume $f$ is defined by $(x, z, p) \mapsto f(x, z, p)$.

**A comment about the admissible perturbations and first variations.** In this case where we have set boundary conditions for $u \in \mathcal{A}$ the set of admissible perturbations is

$$\{\varphi \in X \mid \varphi(a) = \varphi(b) = 0\},$$

since that is the only way we can get $u + \varepsilon\varphi$ to be in $\mathcal{A}$. On the other hand, if we hand not imposed boundary conditions on $u \in \mathcal{A}$, the set of admissible perturbations would just be $X$, since sum and product with a scalar are closed under function (read vector) spaces.

Let's look at how the first variation of a functional of the form of (4.3) looks like.

**Lemma 4.7.** Let $X, \mathcal{A}$ and $\mathcal{I}$ be defined as in (4.3). Then

$$\delta\mathcal{I}(u)(\varphi) = \int_a^b \partial_p f(x, u, u')\varphi'(x) + \partial_z f(x, u, u')\varphi \, dx. \tag{4.4}$$

*Proof.* TODO □

We are almost ready to introduce the Euler-Lagrange theorem, but first we will state an important result in the fields of calculus of variations which will aid us in the proof of the Euler-Lagrange theorem and that will sometimes come handy when checking for the conditions in theorem 4.6.

**Theorem 4.8** (Fundamental theorem of calculus of variations)**.** Let $(a, b) \subset \mathbb{R}$ and $v \in C^0((a, b); \mathbb{R}^m)$ such that

$$\int_a^b v(x) \cdot \varphi(x) \, dx = 0 \text{ for all } \varphi \in C_c^\infty((a, b); \mathbb{R}^m).$$

Then $v = 0$ in $(a, b)$.

As a reminder, in the above theorem $C_c^\infty((a, b); \mathbb{R}^m)$ denotes the set of all infinitely many times continuously differentiable functions with compact support. Also, in this context we are using the closed version of support since $\mathbb{R}$ is a topological space, i.e. we define the support of a function $f : \mathbb{R} \to \mathbb{R}^m$ as the set

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}.$$

Finally, I would like to clarify that the product inside the integral above $v \cdot \varphi$ refers to the standard dot product. One may to thing of the function $v$ as being a row vector and $\varphi$ as a column vector.

*Proof.* See[1] [5, p. 6]. □

Without further ado...

[1] The reason I'm not citing the proof in [1] is that the notation is suboptimal to say the least...

**Theorem 4.9** (Euler-Lagrange theorem)**.** Let $X$ be a functions space, $\mathcal{A} \subset X$ be a set of admissible functions, $\mathcal{I} : X \to \mathbb{R}$ a functional over $x$ and $f \in C^2([a,b] \times \mathbb{R}^m, \times\mathbb{R}^m)$ a real-valued function defined by $(x,z,p) \mapsto f(x,z,p)$. If $u$ is a critical point of $\mathcal{I}$ with $u \in C^2([a,b];\mathbb{R}^m)$ then

$$\frac{d}{dx}\partial_p f(x,u(x),u'(x)) = \partial_z f(x,u(x),u'(x)) \text{ for all } x \in [a,b]. \tag{4.5}$$

In other words, the Euler-Lagrange theorem is giving alternative necessary conditions for $u$ to be a critical point of $\mathcal{I}$.

*Proof.* We use the necessary condition 1 on theorem 4.6 to state $\delta\mathcal{I}(u)(\varphi) = 0$. Now, using (4.4) we write

$$0 = \delta\mathcal{I}(u)(\varphi) = \int_a^b \partial_p f(x,u,u')\varphi'(x) + \partial_z f(x,u,u')\varphi \, dx.$$

The presence of $\varphi'$ is preventing us from applying theorem 4.8 so lets try to get rid of it integrating by parts

$$0 = \cdots = \int_a^b \left[-\frac{d}{dx}\partial_p f(x,u,u') + \partial_z f(x,u,u')\right]\varphi \, dx + \partial_p f(x,u,u')\big|_a^b .$$

Now recall that the boundary conditions for $\varphi$ are $\varphi(a) = \varphi(b) = 0$ and hence the term $\partial_p f\big|_a^b = 0$. Using theorem 4.8 we conclude

$$-\frac{d}{dx}\partial_p f(x,u,u')x + \partial_z f(x,u,u') = 0 \text{ for all } x \in (a,b)$$

and the claim follows. □

An alternative to this proof is to use a more general version of the theorem of calculus of variations which states that if $\int_a^b fh + gh' \, dx = 0$ for every suitable $h$ then $g$ is differentiable and $g' = f$ in $(a,b)$. The version in theorem 4.8 is the special case $g = 0$. For more see [6].

**Remark 4.10.** The previous theorem can be strengthened by only requiring that $u \in C^1$ instead of $C^2$. In this case the du Bois-Reymond lemma[2] is used for the proof in place of theorem 4.8. See [7].

There may be times when the function $f$ only depends on $u$ and $u'$ (read $z$ and $p$) for this we have the following lemma[3].

**Lemma 4.11.** Let $a, b \in \mathbb{R}$ with $a < b$, $u \in C^2((a,b))$ and $f \in C^2(\mathbb{R} \times \mathbb{R})$ such that $\partial_p f(u,u') \in C^1(a,b)$. Then

$$\frac{d}{dx}\partial_p f(u,u') = \partial_z f(u,u') \text{ in } (a,b)$$

[2]Coincidentally, Paul David Gustav du Bois-Reymond (2 December 1831 – 7 April 1889) was just one person.

[3]This lemma originally appeared as a hint for the extra exercise regarding Dido's problem. In that context, it appeared only as a single implication but the proof shows it is really an equivalence.

if, and only if, there exists $c \in \mathbb{R}$ such that

$$f(u,u') - u'\partial_p f(u,u') = c \text{ in } (a,b).$$

*Proof.* Not very precise but something along the lines of

$$\begin{aligned}
&\frac{d}{dx}\partial_p f(u,u') = \partial_z f(u,u') \\
\iff &\partial_z f(u,u') - \frac{d}{dx}\partial_p f(u,u') = 0 \\
\iff &\int \partial_z f(u,u') - \int \frac{d}{dx}\partial_p f(u,u') = \int 0 \\
\iff &f(u,u') - u'\partial_p f(u,u') = c.
\end{aligned}$$

Or, with derivatives,

$$\begin{aligned}
&f(u,u') - u'\partial_p f(u,u') = c \\
\iff &\frac{d}{dx}f(u,u') - \frac{d}{dx}u'\partial_p f(u,u') = \frac{d}{dx}c \\
\iff &\partial_z f(u,u')u' + \partial_p f(u,u')u'' - u''\partial_z f(u,u') - u'\frac{d}{dx}\partial_p f(u,u') = 0 \\
\iff &u'\left(\partial_z f(u,u') - \frac{d}{dx}\partial_p f(u,u')\right) = 0.
\end{aligned}$$

In this case, if $u' \neq 0$ we have

$$\partial_z f(u,u') - \frac{d}{dx}\partial_p f(u,u') = 0 \iff \frac{d}{dx}\partial_p f(u,u') = \partial_z f(u,u').$$

Otherwise, we go back to the original equation

$$f(u,u') - 0\partial_p f(u,u') = c \iff f(u,u') = c$$

which means that $f$ is constant in $u$ and therefore in $x$ so it is trivial to see that

$$\partial_p f(u,u') = 0 = \partial_z f(u,u'),$$

which gives the first equation. □

## 4.2 The direct method

Alright this will be short. The main gain from using the direct method is that one can prove existence of solutions. The form of solutions can be derived by the constructive argument used in the proof.

**Theorem 4.12** (Direct method for variational problems)**.** Let $X$ be a finite-dimensional functions space and consider the variational problem

$$\text{minimise } \mathcal{I}(x) \text{ for } x \in X (= \mathcal{A}).$$

If

1. $\mathcal{I}$ is continuous and
2. $\mathcal{I}$ is coercive, i.e. $\mathcal{I}(x) \to \infty$ as $\|x\|_X \to \infty$,

then the variational problem has a solution.

**Remark 4.13.** The above theorem can be strengthened by requiring that $\mathcal{I}$ is lower semicontinuous instead of continuous from both sides.

## 4.3 Optimal control problems

**Definition 4.14** (Optimal control problem)**.** We define an optimal control problem as the task

$$\text{maximise } \mathcal{I}(y,u) = g(y(T)) + \int_0^T r(y(t),u(t))\,dt \text{ for } u \text{ and } y \tag{4.6}$$

such that

$$\begin{cases} y'(t) &= f(y(t),u(t)) \text{ in } [0,T] \\ y(0) &= y_0 \end{cases} \tag{4.7}$$

where

- $y : [0,T] \to \mathbb{R}^n$ is the state variable (for some $T > 0$),
- $y_0 \in \mathbb{R}^n$ is the initial state, and
- $u : [0,T] \to A \subset \mathbb{R}^n$ is the control variable,

given

- the cost function $\mathcal{I} : \mathbb{R}^n \times A \to \mathbb{R}$,
- a function $r : \mathbb{R}^n \times A \to \mathbb{R}$ defined by $(w,z) \mapsto r(w,z)$ so that $\partial_w r, \partial_z r \in \mathbb{R}^n$,
- a function $g : \mathbb{R}^n \to \mathbb{R}$, and
- a function $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ defined by $(w,t) \mapsto f(w,z)$.

Long story short to tackle this beast we have

**Theorem 4.15** (Pontryagin's maximum principle)**.** Let $(\overline{y}, \overline{y})$ be a solution to (4.6). Then there exists a function $\overline{p} : [0, T] \to \mathbb{R}^n$ called the adjoin state such that in $[0, T]$ the following hold

1. $\overline{y}' = \partial_p H(\overline{y}, \overline{u}, \overline{p}),\ \overline{y}(0) = y_0$,
2. $\overline{p}' = -\partial_w H(\overline{y}, \overline{u}, \overline{p}).\ \overline{p}(T) = \nabla g(\overline{y}(T))$, and
3. $H(\overline{y}, \overline{u}, \overline{p}) = \max_{z \in A} H(\overline{y}, z, \overline{p})$,

where

$$H(w, z, q) = f(w, z) \cdot q + r(w, z) \text{ for } w, q \in \mathbb{R}^n \text{ and } z \in A \subset \mathbb{R}^n.$$

Furthermore, the map $t \mapsto H(\overline{y}(t), \overline{u}(t), \overline{p}(t))$ is constant.

*Proof.* hahahahahahahhahahahahahahahah □

## 4.4 Exercises

**Exercise** (Dido's problem)**.** Let $L > 0$ be a given length. We consider the maximisation problem

$$\text{maximise } \int_0^L u(s)\sqrt{1 - u'(s)^2}ds, \text{ for } u \in \mathcal{A},$$

where $\mathcal{A} = \{C^1((0, L)) \cap C^0([0, L]) : u(0) = 0, u(L) = 0, |u'(s)| \text{ for } s \in (0, L)\}$, which emerges from modeling Dido's problem.

1. Determine the corresponding Euler-Lagrange equation and find a non-negative solution $\overline{u} \in \mathcal{A} \cap C^2((0, L))$.
2. Sketch the curve $\{(\varphi(s), \overline{u}(s)) : s \in [0, L]\}$ with $\varphi(s) = \int_0^s \sqrt{1 - \overline{u}'(\tau)}d\tau$ for $s \in [0, L]$.
3. Interpret b) in the context of Dido's problem.

*Proof.* For the purposes of determining the Euler-Lagrange equation we have $x = s,\ z = u(x),\ p = u'(s)$ and $f(x, z, p) = z\sqrt{1 - p^2}$. The involved partial derivatives are

$$\partial_z f(x, z, p) = \sqrt{1 - p^2} \text{ and } \partial_p f(x, z, p) = -\frac{zp}{\sqrt{1 - p^2}},$$

and thus the Euler-Lagrange equation becomes

$$\frac{d}{dx}\frac{-uu'}{\sqrt{1 - u'^2}} = \sqrt{1 - u'^2}.$$

This looks complicated so we apply the lemma 4.11:

$$\begin{aligned}
& u\sqrt{1-u'^2} + u'\frac{uu'}{\sqrt{1-u'^2}} = c \\
\iff & u\left(\sqrt{1-u'^2} + \frac{u'^2}{\sqrt{1-u'^2}}\right) = c \\
\iff & u\left(\frac{1-u'^2+u'^2}{\sqrt{1-u'^2}}\right) = c \\
\iff & u = c\sqrt{1-u'^2} \\
\iff & u' = \sqrt{1-\left(\frac{u}{c}\right)^2}.
\end{aligned}$$

That last equation desperately screams for separation of variables and a trigonometric change of variable:

$$\begin{aligned}
& \frac{du}{ds} = \sqrt{1-\left(\frac{u}{c}\right)^2} \\
\iff & \frac{du}{\sqrt{1-(u/c)^2}} = ds \\
\iff & \int \frac{1}{\sqrt{1-(u/c)^2}} du = \int ds.
\end{aligned}$$

Change $u/c = \sin y \implies u = c\sin y \implies du = c\cos y dy$ to get

$$\begin{aligned}
\int \frac{1}{\sqrt{1-(u/c)^2}} du &= \int \frac{1}{1-\sin^2 y} c\cos y dy \\
&= c\int \frac{\cos y}{\cos y} dy = cy \\
&= c\arcsin\frac{u}{c}
\end{aligned}$$

Plugging it back into the hint,

$$s + k = c\arcsin\frac{u}{c} \implies u(s) = c\sin\frac{k+s}{c},$$

which we rewrite picking different constants,

$$u(s) = k_1\sin(k_2 s + k_3),$$

where the constants $k_1, k_2, k_3$ are obtained by enforcing $u \in \mathcal{A}$. More specifically, we require

$$u(0) = 0 \implies k_3 = 0 \text{ and } u(L) = 0 \implies k_2 L = n\pi.$$

Moreover, we require $u$ to be non-negative, therefore $n = 1$ and thus $k_2 = \frac{\pi}{L}$ (otherwise the sine would go negative). Finally, we want

$$|u'(s)| < 1 \implies |k_1\cos(k_2 s + k_3)k_2| < 1 \implies k_1 k_2 < 1 \implies k_1 < \frac{L}{\pi},$$

since we require that $k_1$ is positive so that $u(s)$ also is. This final parameter is fixed by maximising $\mathcal{I}(u)$:

TODO □

**Exercise** (Geodesics in $\mathbb{R}^2$)**.** Let $A$ and $B$ be two points in the plane. What is the shortest connection between $A$ and $B$?

1. Set up the variational problem to model the situation.
2. Solve the problem and interpret the result.

*Proof.* Let $X = C^1([0,1];\mathbb{R}^2)$ and $\mathcal{A} = \{u \in X \mid u(0) = A,\ u(1) = B\}$. We define our functional $\mathcal{I}$ as the length of the parametrised curve $u$ as follows

$$\mathcal{I}(u) = \int_0^1 \|u'(t)\|\, dt.$$

Our variational problem is

$$\text{minimise } \mathcal{I}(u) \text{ for } u \in \mathcal{A}.$$

To solve it we use the Euler-Langrange method. We have $f(t, u(t), u'(t) = \|u'(t)\|$ and, in the form of the Euler-Langrange equations, we get $f(x, z, p) = \|p\|$. Therefore,

$$\delta_z f = \begin{pmatrix} 0 & 0 \end{pmatrix} \text{ and } \delta_p f = \begin{pmatrix} \frac{p_1}{\|p\|} & \frac{p_2}{\|p\|} \end{pmatrix}.$$

Notice how $p = u' : [0,1] \to \mathbb{R}^2$ so by $\delta_p f$ we really mean the last two numbers in $Df$ (which is a row matrix since $f$ is real valued). We arrive at the following Euler-Lagrange equation

$$\frac{d}{dt}\delta_p f = \delta_z \iff \frac{d}{dt}\begin{pmatrix} \frac{u_1}{\|u\|} & \frac{u_2}{\|u\|} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Instead of taking the derivative with respect to $t$, we may simply rewrite this as

$$\begin{pmatrix} \frac{u_1}{\|u\|} & \frac{u_2}{\|u\|} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{R}$ are constants.

Therefore

$$u(t) = \int u'(t)dt = (c_1 t, c_2 t) + u_0,\ u_0 \in \mathbb{R}^2,$$

which is a parametrisation for a curve. The parameters $c_1, c_2$ and $u_0$ are determined by enforcing $u \in \mathcal{A}$:

$$u(0) = u_0 = A,\ u(1) = (c_1, c_2) + u_0 = B.$$

□

# Chapter 5

# Continuum mechanics

This chapter deals with the physics of a very general body that we describe using calculus, hence the name continuum. The main assumption that we make to be able to do this is that the behaviour of a system can be described by averaging some characteristic quantities over the space it occupies, instead of computing every one of them for every atom or molecule, i.e. that the involved quantities in a problem are defined in a continuum. It turns out this assumption is pretty good and the models we will get closely describe what is happening in reality.

## 5.1 Frames of reference and coordinate systems

Before we begin we need a way to describe points in space. We will do this using the standard definitions and results of affine spaces that we get from linear algebra.

**Definition 5.1.** Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for $R^n$ and let $O \in \mathbb{R}^n$ be a point which we will call the origin. In the equation

$$X = O + \sum_{i=1}^{n} x_i e_i,$$

we define $(x_1, \dots, x_n)$ to be the coordinates of the point $X$ with respect to the origin $O$.

Notice how in the above definition we talked about points and not vectors when dealing with $X$ and $O$. The distinction is rather technical and can almost always be ignored but in short the deal is that points live in the affine space $\mathbb{A}^n_{\mathbb{R}} = (A, \mathbb{R}^n, f)$ while vectors live in $\mathbb{R}^n$. An affine space always comes with a function that maps points to vectors and is generally of the form

$$f : A \times A \to \mathbb{R}^n, \qquad (P, Q) \mapsto \overrightarrow{PQ},$$

where $\overrightarrow{PQ}$ denotes the vector which starts at $P$ and ends at $Q$ that can be computed by treating $P$ and $Q$ as vectors in $\mathbb{R}^n$ and taking the difference. More details can be found in [8].

One important takeaway is the following lemma.

**Lemma 5.2.** Let $(O; e_1, \dots, e_n)$ and $(O^*, e_1^*, \dots, e_n^*)$ be two affine coordinate systems and let $X$ be a point in $\mathbb{A}^n_{\mathbb{R}}$ defined in the two coordinate systems, i.e.

$$X = O + \sum_{i=1}^{n} x_i e_i = O^* + \sum_{i=1}^{n} x_i^* e_i^*,$$

then there exists $a \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ orthogonal such that

$$x^* = a + Qx.$$

## 5.2 Introduction and setup of continuum mechanics

### 5.2.1 Lagrangian and Eulerian coordinate systems

Some of the problems we will focus on require us to describe the evolution of the shape of a body. We might do this by following the position of a point inside the body as time passes. To be precise, we choose an initial state or **reference configuration** $\Omega \subset \mathbb{R}^n$, which is normally the set of all points in space which are part of the body under consideration. Lets say that **material point** is called $X \in \Omega$. We can think of $X$ as a way to pinpoint an atom in the body and lets assume that we can track it as time passes. We might describe its position with the function

$$x : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \qquad (t, X) \mapsto x(t, X).$$

On the other hand, we may be interested in some other quantities, for instance the temperature at a point in the body. While it could be the case that we are interested in the temperature of a particular atom (even if that made sense), it is most likely that we are looking for the evolution of temperature at a fixed point in space. It could even be that we are dealing with a fluid and looking at individual atoms or molecules does not make sense. In both of this cases we are interested in a quantity $\varphi(t, x)$ where $x \in \mathbb{R}^n$. The choice of the letter $x$ for the physical point is not a coincidence since for a material point $X$ in the reference configuration we can get its position at time $t$ with $x(t, X)$ and therefore $\varphi(t, x) = \varphi(t, x(t, X))$.

Let us formalise these concepts. We start by making suitable assumptions

1. The material point $X$ is described by its position at some initial time $t = t_0$ by the function $X = x(t_0, X)$.

2. The mapping $(t, X) \mapsto x(t, X)$ is continuously differentiable.

3. For all $t \geq t_0$, the mapping $x$ is invertible.

4. $x$ is orientation preserving, i.e. the matrix[1]

$$Jx(t,X)_{ij} = \left(\frac{\partial x_i}{\partial X_j}(t,X)\right)$$

has a positive determinant for every $t \geq t_0$ and every $X \in \Omega$.

The terms $x$ and $X$ represent two different types of coordinates[2] corresponding to the two ways of looking at a problem described before:

- **Lagrangian coordinates** $X$ allow us to choose one material point and follow its evolution, while

- **Eulerian coordinates** $x$ relate to a fixed point in space.

In general, we will observe different material points at different times at point $x$ specified in Eulerian coordinates.

### 5.2.2 Analysis of relevant quantities in Lagrangian and Eulerian coordinate systems

Before we move on, let's take a moment to really formalise how the function $x$ is defined and to look at how we might take the derivatives. In the book ([1, p. 208]) we can find the following definition

**Definition 5.3** (Material time derivative)**.** Let $\varphi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. We define the material time derivative of $\varphi$ with respect to $t$ as

$$D_t\varphi(t,x) = \partial_t\varphi(t,x) + \nabla\varphi(t,x) \cdot v(t,x),$$

where $v(t,x) = \partial_t x(t, X(t,x))$.

To me, this did not make sense at all on the first reading. It also didn't make sense on the second, third, and $n$-th readings, for a substantially large $n$. What helped me understand it was to not take it as a definition, but to try to arrive at the same result with standard vector calculus results, i.e. the chain rule. In what follows I try to describe this to the best of my ability.

Take any $X_0 \in \Omega$ and define $x_0(t) = x(t, X_0)$, i.e. fix the second parameter of the function $x(t,X)$ so that it describes the position of only one particular material point $X_0$. In this case,

$$\frac{d}{dt}x_0(t) = \partial_t x(t,X)|_{X=X_0}.$$

[1] The book [1, p. 207] calls the determinant of this matrix the Jacobi determinant. In reality, $Jx$ is just the the differential matrix $Dx$ only we omit the first row since we are interested in the derivatives with respect to the coordinates of the material point $X$. Notice how we say differential matrix since the Jacobian matrix is guaranteed to exist and be the derivative because we asked for $x$ to be continuously differentiable.

[2] I tried to phrase this first part of the section as a standard definition-properties introductory section. However, these constructs are sometimes just notational conventions used by people who work with different branches of continuum mechanics and hence are not normally that clearly defined. One may take this section with a grain of salt and just make sure the math makes sense when dealing with equations on each coordinate system.

Now going back to our relevant quantity $\varphi$ we do the same thing—restrict it to work for just one material point $X_0$. In this case $\varphi$ becomes $\varphi(t, x_0(t))$.

We will use the convention that quantities expressed as a function of Lagrangian coordinates will be denoted by [greek] capital letters like $\Phi(t, X)$ while quantities defined on Eulerian coordinates will be identified by a lowercase letter $\varphi(t, x(t, X))$. It is important to observe that here

$$\Phi(t, X) = \varphi(t, x(t, X)),$$

that is, they denote the same quantity and hence evaluate to the same mathematical object (in some cases we use vectors or matrices instead of just numbers to represent relevant quantities).

One instance where this convention is used is when dealing with the velocity fields, which are

$$V(t, X) = \frac{\partial x}{\partial t}(t, X)$$

in Lagrangian coordinates, and

$$v(t, x) = V(t, X(t, x))$$

in Eulerian coordinates. Here $X$ is the inverse of the function $x$ which we assumed to exist for $t \geq t_0$.

More generally, we may wish to calculate the derivative of a quantity. In Eulerian coordinates, when we denote a quantity by $\varphi(t, x)$ we must not forget the fact that $x$ is itself a function of $t$ and a material point $X$. More explicitly, we may write $\varphi(t, x(t, X))$ as before to make it easier to compute the derivative with respect to time using the chain rule.

We conclude this section by defining two sets of important curves, namely pathlines and streamlines.

**Definition 5.4** (Pathlines)**.** Pathlines are solutions of the equation

$$y'(t) = v(t, y(t)).$$

They describe the paths followed by a material point during the evolution of the system.

**Definition 5.5** (Streamlines)**.** Streamlines are solutions to the equation

$$z'(s) = v(t, z(s)).$$

They are curves tangent to the velocity field at every point, i.e. they give a *snapshot* of the velocity field at time $t$.

Observe that in streamlines we treat $t$ as a parameter whereas the equation for pathlines is more involved. See the solution TODO add reference to extra problem.

Throughout the next sections, the structure will be the following. We will first focus on fluids and for that we will use Eulerian coordinates. This will take the greater part of the chapter. Finally, we will conclude with elastic solids, for which we will use Lagrangian coordinates.

## 5.3 Reynold's transport theorem

We start with some basic results from calculus / algebra.

**Lemma 5.6** (Jacobi's formula)**.** Let $t \mapsto A(t) \in \mathbb{R}^{n\times n}$ be a differentiable map such that $A(t)$ is invertible. Then,

$$\frac{d}{dt}\det A(t) = \operatorname{tr}\left(A^{-1}(t)\frac{d}{dt}A(t)\right)\det A(t).$$

This result can be proved by setting $\det A = F(a_{11},\dots,a_{nn})$ as a function of the entries of the matrix $A$ and appyling the cain rule with a lot of care. See TODO for details.

**Theorem 5.7.** Let $(t,X) \mapsto x(t,X)$ be a continuously differentiable, invertible and orientation preserving mapping, and that $(t,X) \mapsto \partial_t x(t,X)$ is also continuously differentiable. Then,

$$\partial_t J(t,X) = \nabla \cdot v(t,x)|_{x=x(t,X)}\, J(t,X),$$

where $J$ is the matrix introduced in the previous section.

# Chapter 6

# Partial differential equations

Unlike with ODEs, there is no general theorem for existence, uniqueness or structure of solutions for PDEs. In this section we will classify some of the most common PDEs and try to give some results for each category that help solve them or gain some insight on how the solutions behave.

**Definition 6.1.** A partial differential equation or PDE is an identity of the form

$$F(D^k, \dots, D^2, \nabla u, u, x) = 0 \tag{6.1}$$

where

- $x \in \Omega \subset \mathbb{R}^n$ is the independent variable,
- $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is the function we want to solve (6.1) for,
- TODO

TODO: comment about the order

## 6.1 First-order partial differential equations

In general, first-order PDEs can be linear, semilinear, quasilinear or non-linear. An example of a non-linear equation is the TODO eq

$$\|\nabla u(x)\| = 1.$$

An example of a linear equation is the generic transport equation

$$\partial_t u(t, x) + \partial_x u(t, x) = 0.$$

In general all but non-linear equations can be solved using the so called **method of characteristics** which exploits certain geometric properties to reduce the PDE to a system of ODEs.

### 6.1.1 Method of characteristics for linear partial differential equations

## 6.2 Second-order partial differential equations

In this harder case, we will restrict ourselves to just linear equations.

**Definition 6.2** (Second-order, linear PDE)**.** We say the identity

$$\sum_{j,k=1}^{n} a_{jk}(x)\partial_{x_j}\partial_{x_k}u(x) + F(\nabla u(x), u(x), x) = 0 \tag{6.2}$$

with

$$A(x) = (a_{jk}(x))_{j,k=1}^{n} \in \mathbb{R}^{n\times n} \text{ symmetric}$$

is a linear second-order partial differential equation.

Note that the symmetricness of the matrix $A(x)$ comes from Schwarz's theorem for mixed derivatives.

**Definition 6.3.** Consider a semilinear second-order PDE of the form of (6.2). For any $x \in \Omega$ we say (6.2) is

- **elliptic** at $x$ if all the eigenvalues of $A(x)$ are non-zero and have the same sign,
- **parabolic** at $x$ if $A(x)$ has the eigenvalue 0 with multiplicity one and all other eigenvalues have the same sign, and
- **hyperbolic** at $x$ if 0 is not an eigenvalue of $A(x)$ and exactly $n-1$ eigenvalues have the same sign.

If any of the previous contitions holds for every $x \in \Omega$ we say (6.2) is **globally** elliptic, resp. parabolic, hyperbolic.

**Example 6.4.** The Laplace equation $\Delta u(x) = 0$ is globally elliptic since

$$\Delta u(x) = \sum_{i=1}^{n} \partial_{x_i}^2 u(x) = \sum_{i=1}^{n} 1\partial_{x_i}\partial_{x_i}u(x) \implies A(x) = \mathrm{Id}.$$

**Example 6.5.** The heat equation $\partial_t u(t,x) - \Delta u(t,x) = 0$ is globally parabolic since

$$A(x) = \left(\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & -1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & -1 \end{array}\right).$$

**Example 6.6.** The wave equation $\partial_t^2 u(t,x) - \Delta u(t,x) = 0$ is globally hyperbolic since

$$A(x) = \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & -1 & \dots & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & -1 \end{array}\right).$$

### 6.2.1 Results for elliptic second-order partial differential equations

Since the elliptic is the simplest of the three categories, we will use the remaining time to give some results about two very important examples of these equations: the Laplace equation and the Poisson equation.

#### Fundamental solution to the Laplace equation

Recall that the Laplace equation is of the form

$$\Delta u(x) = 0 \tag{6.3}$$

for some $u$ TODO.

We make the ansatz $u(x) = w(\|x\|) = w(r)$, i.e. we assume rotational symmetry of the solution. If this were the case then (6.3) becomes

$$\Delta u$$

**Definition 6.7** (Convolution product)**.** Given two functions $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ we define the convolution product of $f$ and $g$ as

$$(f * g)(x) = \int_\Omega f(x-y)g(y)dy.$$

That we name this weird thing product is not coincidental, as it shares some properties with the standard number product.

**Theorem 6.8** (Properties of the convolution product)**.** Let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be two functions. Then

1. (Conmutativity) $f * g = g * f$, and
2. (Linearity with respect to differentiation)

$$\partial_{x_j}^k (f * g) = (\partial_{x_j}^k f) * g = f * (\partial_{x_j}^k g).$$

We will finish the course with two results about vector valued functions.

**Theorem 6.9.** Let $\Omega \subset \mathbb{R}^n$ and let $B_R(x) \subset \Omega$ be a ball with radius $R > 0$ centered in $x \in \Omega$. If $u \in C^2(\Omega)$ is harmonic then

$$u(x) = \frac{1}{\|\partial B_R(x)\|} \int_{\partial B_R(x)} u(y) ds_y = \frac{1}{\|B_R(x)\|} \int_{B_R(x)} u(y) dy,$$

where $\|\partial B_R(x)\|$ denotes the hypersurface area of the boundary of $B_R(x)$ and $\|B_R(x)\|$ denotes the hypervolume of the ball $B_R(x)$.

Note that in the previous theorem we have not inlined the well known formulas for the area and volume of a sphere as these only hold for $R^3$.

TODO: other theorem or maximum principle.

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